Catalan Numbers
CS 491 – Competitive Programming

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Spring 2023

Objectives

▶ Compute the nth Catalan number

▶ Map the Catalan numbers to various isomorphisms.

Example 1: Parenthesis

▶ Suppose you have a $2n$ parenthesis characters, half open, half closed. How many "algebraically legal" sequences are there?

  ▶ Every open parenthesis followed by a matching close parenthesis.
  ▶ No close parenthesis before its corresponding open.

▶ $n = 2$

  ▶ ()()
  ▶ (())

▶ $n = 3$

  ▶ ((()))
  ▶ ()(())
  ▶ ()()()
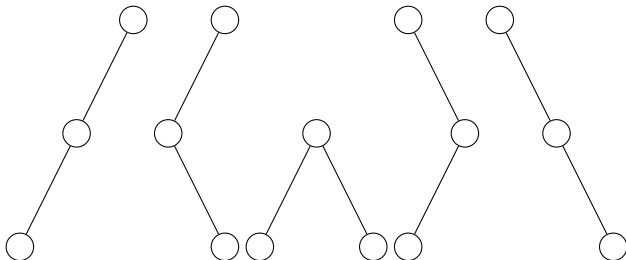 ▶ (())()
  ▶ (()())

▶ For $n = 4$ there are 14.

Example 2: Trees

▶ Given n indistinguishable nodes, how many binary trees can you make?
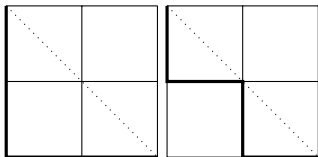
▶ $n = 2$
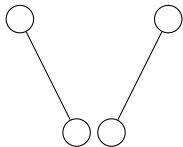
▶ $n = 3$

Example 3: Integer Sequences of +1 and -1

- ▶ Suppose you have *n* copies of 1 and *n* copies of $-$.
- ▶ How many sequences of $a_0, a_1, ... a_{2n}$ are there such that for all $k < 2n$, $0 \leq \Sigma_{i=0}^{2k} a_i$?
- ▶ $n = 2$
  - ▶ 1,1,-1,-1
  - ▶ 1,-1,1,-1
- ▶ $n = 3$
  - ▶ 1,-1,1,-1,1,-1
  - ▶ 1,-1,1,1,-1,-1
  - ▶ 1,1,-1,-1,1,-1
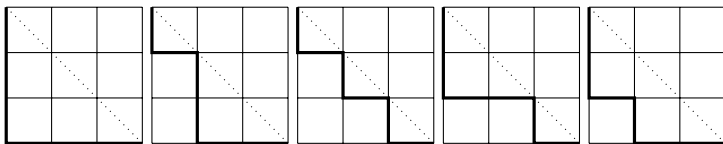 - ▶ 1,1,-1,1,-1,-1
  - ▶ 1,1,1,-1,-1,-1

Example 4: Matrix Walk

▶ Suppose you have a $n \times n$ matrix. Start from the top left and reach the bottom right without crossing the diagonal.

▶ $n = 2$

▶ $n = 3$

The Formula

- Recursively: $Can(n) = \frac{(2n+1)(2n)}{(n+1)n} Cat(n-1)$
  - $Cat(0) = 1, Cat(1) = 1, Cat(2) = 2, Cat(3) = 5, Cat(4) = 14, ...$
- Others things:
  - number of ways to triangulate a polygon
  - Number of ways to count a tie vote so that candidate A never passes candedate B